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A GENERALIZATION OF THE CLASSICAL PARTITION THEOREMS

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1. **Introduction.** Let $P_d(n)$ denote the number of partitions of n into positive integers of the form $n=b_1+\cdots+b_s$ with $b_i-b_{i+1}\geqq d$ where strict inequality holds if $d|b_i$. $P_1(n)$ appears in the first Rogers-Ramanujan identity [5, p. 291]. $P_2(n)$ appears in the first of the Göllnitz-Gordon partition theorems [2, p. 945], and $P_3(n)$ appears in a partition theorem of I. J. Schur [6, p. 489].

For $d>3$, we have the following result due to H. L. Alder [1, p. 713].

(1.1) Let S be any fixed set of positive integers. Then $P_d(n)$ is not identically equal to the number of partitions of n into parts taken from S if $d>3$.

In a different approach, B. Gordon [4, p. 394] has made the following generalization of the Rogers-Ramanujan identities.

(1.2) Let a and k be integers with $0<a\leqq k$. Let $A_{k,a}(n)$ denote the number of partitions of n into parts $\not\equiv 0$, $\pm a \pmod{2k+1}$. Let $B_{k,a}(n)$ denote the number of partitions of n of the form $n=b_1+\cdots+b_s$ with $b_i\geqq b_{i+1}$, $b_i-b_{i+k-1}\geqq 2$, and with 1 appearing as a summand at most $a-1$ times. Then $A_{k,a}(n)=B_{k,a}(n)$.

The Göllnitz-Gordon identities have also been generalized in this manner [2, p. 946].

(1.3) Let a and k be integers with $0<a\leqq k$. Let $A^{\#}_{k,a}(n)$ denote the number of partitions of n into parts which are neither $\equiv 2 \pmod 4$ nor $\equiv 0$, $\pm(2a-1) \pmod{4k}$. Let $B^{\#}_{k,a}(n)$ denote the number of partitions of n of the form $n=b_1+\cdots+b_s$, with $b_i\geqq b_{i+1}$, no odd parts are repeated, $b_i-b_{i+k-1}\geqq 2$ with strict inequality if $2|b_i$ and at most $a-1$ parts not exceeding 2 appear. Then $A^{\#}_{k,a}(n)=B^{\#}_{k,a}(n)$.

This leads us to define the following partition function. Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $n=b_1+\cdots+b_s$, with $b_i\geqq b_{i+1}$, no parts $\not\equiv 0 \pmod{\lambda+1}$ are repeated, $b_i-b_{i+k-1}\geqq \lambda+1$ with strict inequality if $\lambda+1|b_i$ and at most $a-1$ parts not exceeding $\lambda+1$ appear. We note that $B_{0,k,a}(n)$ appears in (1.2); $B_{1,k,a}(n)$ appears in (1.3), and $B_{d-1,2,2}(n)=P_d(n)$.

QUESTION. *If k is sufficiently large with respect to λ, does $B_{\lambda,k,a}(n)$ appear in a partition theorem of the Rogers-Ramanujan type?*

We answer this question affirmatively with the following result.

THEOREM 1. *Let $0\leqq\lambda\leqq a\leqq k$ be integers, and $2\lambda-1\leqq k$. If λ is even, define*

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$A_{\lambda,k,a}(n)$ to be the number of partitions of n into parts such that no part $\not\equiv 0 \pmod{\lambda+1}$ *may be repeated and no part is* $\equiv 0, \pm(a-\frac{1}{2}\lambda)(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}$. *If* λ *is odd, define $A_{\lambda,k,a}(n)$ to be the number of partitions of n into parts such that no part* $\not\equiv 0 \pmod{\frac{1}{2}(\lambda+1)}$ *may be repeated, no part is* $\equiv \lambda+1 \pmod{2\lambda+2}$, *and no part is* $\equiv 0, \pm(2a-\lambda)\frac{1}{2}(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}$. *Then* $A_{\lambda,k,a}(n)=B_{\lambda,k,a}(n)$.

There is numerical evidence to suggest that the condition "$k \geqq 2\lambda-1$" is unnecessary; but, the proof given here strongly relies on this condition. Actually the case $k=\lambda=a=2$ violates this condition; however, Theorem 1 becomes Schur's theorem in this case.

In §2, we shall develop results concerning auxiliary partition functions. In §3, we shall prove Theorem 1; our work will be based on the generalized well-poised basic hypergeometric functions studied in [3]. In §4, we prove a more general theorem than Theorem 1 in which the parameter a is allowed to take values in the interval $(\frac{1}{2}\lambda, k]$. In §5, we discuss possibilities for further work.

2. **Auxiliary partition functions.** Let $\pi(k, \lambda, t, j, \mu; n)$ denote the number of partitions of n of the form $n=\sum_{i=1}^{2\lambda+1} f_i \cdot i$ (here f_i is the number of times the summand i appears) where

(2.1) $f_1+\cdots+f_{\lambda+1}=k-\lambda+j$;

(2.2) $f_{\lambda+1}+\cdots+f_{2\lambda+1}=k-t$;

(2.3) $f_m+\cdots+f_{m+\lambda} \geqq k$ for some m;

(2.4) $f_a>1$ implies $a=\lambda+1$;

(2.5) $f_1+\cdots+f_{2\lambda+1}=k+\mu$.

Throughout our discussion we shall assume k, λ, t, j, and μ are all nonnegative integers with $j \leqq \lambda$, $k \geqq \max(1, 2\lambda-1)$, and $\pi(k, \lambda, t, j, \mu; 0)=0$.

Let $g(k, \lambda, t, j, \mu; q)=\sum_{n=0}^{\infty} \pi(k, \lambda, t, j, \mu; n)q^n$.

LEMMA 2.1.

$$g(k, \lambda, t, j, \mu; q) = q^{(k-t)(\lambda+1)}\sigma_\mu(\lambda)\sigma_{\lambda+\mu+t-j}(\lambda), \tag{2.6}$$

where

$$\begin{aligned} \sigma_i(\lambda) &= q^{i(i+1)/2}\begin{bmatrix}\lambda\\ i\end{bmatrix}, \\ &= q^{i(i+1)/2}\frac{(q^\lambda-1)\cdots(q^{\lambda-i+1}-1)}{(q^i-1)\cdots(q-1)}, \qquad 0 \leqq i \leqq \lambda, \\ &= 0, \qquad \textit{otherwise}; \end{aligned}$$

thus $\sigma_i(\lambda)$ is just the generating function for partitions into exactly i distinct parts with all parts $\leqq \lambda$.

Proof. We proceed by induction on the second argument of g, namely λ. If $\lambda=0$, then since $j \leqq \lambda$, $j=0$. The conditions (2.1)–(2.5) imply $f_1=k+j=k-t$

$=k+\mu \geqq k$; thus $j=t=\mu=0$. Thus the only allowable partition is of k into k "ones". Therefore

$$\begin{aligned} g(k, 0, t, j, \mu; q) &= q^k \quad \text{if } \mu = j = t = 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

On the other hand, consider the right-hand side of (2.6); call it $G(k, 0, t, j, \mu; q)$. The only way $G(k, 0, t, j, \mu; q)$ will be nonzero is if $\mu=0$, $t=j$, and since $j \leqq \lambda$ we have $t=j=0$. Hence

$$\begin{aligned} G(k, 0, t, j, \mu; q) &= q^k \quad \text{if } \mu = j = t = 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Thus the lemma holds for $\lambda=0$.

If $\mu=0$, there are just k parts appearing in the partition by (2.5), and since (2.3) must hold we see that there exists a particular m such that

$$f_m + \cdots + f_{m+\lambda} = k, \qquad f_i = 0 \quad \text{for } i < m \text{ or } i > m+\lambda.$$

If we set $A=f_1+\cdots+f_\lambda$, $B=f_{\lambda+2}+\cdots+f_{2\lambda+1}$, then (2.1), (2.2), and (2.5) imply $A=t+\mu$, $f_{\lambda+1}=k+j-\lambda-t-\mu$, $B=\lambda+\mu-j$; thus in case $\mu=0$, $A=t$, $f_{\lambda+1}=k+j-\lambda-t$, $B=\lambda-j$. Let us now consider any partition of the type described in this paragraph. If we subtract $\lambda+1$ from each of the $\lambda-j$ parts greater than $\lambda+1$, we get a partition with $\lambda+1$ appearing $k+j-\lambda-t$ times with $\lambda+t-j$ remaining parts which are distinct and are all $\leqq \lambda$. Conversely if we have a given partition in which $\lambda+1$ appears $k+j-\lambda-t$ times and there are $\lambda+t-j$ remaining parts which are distinct and are $\leqq \lambda$, we define m to be the integer such that $f_1+\cdots+f_{m-1}=\lambda-j$, and we add $\lambda+1$ to each part $<m$; this gives us precisely the type of partition described at the beginning of this paragraph. Hence by the definition of $\sigma_j(\lambda)$,

$$\begin{aligned} g(k, \lambda, t, j, 0; q) &= q^{(k+j-\lambda-t)(\lambda+1)+(\lambda-j)(\lambda+1)}\sigma_{\lambda+t-j}(\lambda) \\ &= q^{(k-t)(\lambda+1)}\sigma_{\lambda+t-j}(\lambda) \\ &= G(k, \lambda, t, j, 0; q), \end{aligned}$$

and (2.6) is verified in this case.

We now proceed by induction on λ (>0), and we may also assume $\mu>0$. Our first task is to verify

$$\begin{aligned} \pi(k, \lambda, t, j, \mu; n) = {} & \pi(k, \lambda-1, t, j-1, \mu; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j, \mu; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j-2, \mu-1; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j-1, \mu-1; n+j-k-2\mu-\lambda). \end{aligned} \tag{2.7}$$

To verify (2.7) we proceed as follows. Let us consider a partition of n of the type enumerated by $\pi(k, \lambda, t, j, \mu; n)$. Let us subtract 1 from every summand $\leqq \lambda+1$, and let us subtract 2 from each summand $>\lambda+1$. The number being partitioned now is $n-(k+j-\lambda)-2(\lambda+\mu-j)=n+j-k-2\mu-\lambda$. Four cases are to be considered depending on: (1) $f_1=f_{\lambda+2}=0$; (2) $f_1=0, f_{\lambda+2}=1$; (3) $f_1=1, f_{\lambda+2}=0$; (4) $f_1=f_{\lambda+2}=1$.

We let f_i' denote the number of appearances of i in the transformed partition. Thus for example in case (1), conditions (2.1)–(2.5) now read

$(2.1)_1$ $f_1'+\cdots+f_\lambda'=k-(\lambda-1)+j-1$;

$(2.2)_1$ $f_\lambda'+\cdots+f_{2\lambda-1}'=k-t$;

$(2.3)_1$ $f_m'+\cdots+f_{m+\lambda-1}'\geqq k$ for some m;

$(2.4)_1$ $f_a'>1$ implies $a=\lambda$;

$(2.5)_1$ $f_1'+\cdots+f_{2\lambda-1}'=k+\mu$.

Thus we have a partition of the type enumerated by

$$\pi(k, \lambda-1, t, j-1, \mu; n+j-k-2\mu-\lambda).$$

Indeed the above procedure establishes a one-to-one correspondence between those partitions enumerated by $\pi(k, \lambda, t, j, \mu; n)$ which fall in case (1) and the partitions enumerated by $\pi(k, \lambda-1, t, j-1, \mu; n+j-k-2\mu-\lambda)$. Similarly in the other three cases we get the other three partition functions appearing on the right-hand side of equation (2.7). Thus (2.7) is established.

(2.7) implies

$$\begin{aligned} g(k, \lambda, t, j, \mu; q) = q^{k+\lambda+2\mu-j}(&g(k, \lambda-1, t, j-1, \mu; q)+g(k, \lambda-1, t, j, \mu; q)\\ &+g(k, \lambda-1, t, j-2, \mu-1; q)\\ &+g(k, \lambda-1, t, j-1, \mu-1; q)). \end{aligned} \tag{2.8}$$

Let us now examine the right-hand side of (2.6) which we have called $G(k, \lambda, t, j, \mu; q)$. Now

$$\sigma_j(\lambda) = q^j\sigma_j(\lambda-1)+q^j\sigma_{j-1}(\lambda-1); \tag{2.9}$$

this is merely a restatement of the well-known formula for Gaussian polynomials

$$\begin{bmatrix}\lambda\\ j\end{bmatrix} = \begin{bmatrix}\lambda-1\\ j-1\end{bmatrix}+q^j\begin{bmatrix}\lambda-1\\ j\end{bmatrix}. \tag{2.10}$$

Consequently

$$\begin{aligned} G(k, \lambda, t, j, \mu; q) &= q^{(k-t)(\lambda+1)}\sigma_\mu(\lambda)\sigma_{\lambda+\mu+t-j}(\lambda)\\ &= q^{k-t+\mu+\lambda+\mu+t-j}q^{(k-t)\lambda}(\sigma_\mu(\lambda-1)+\sigma_{\mu-1}(\lambda-1))\\ &\quad\times(\sigma_{\lambda+\mu+t-j}(\lambda-1)+\sigma_{\lambda+\mu+t-j-1}(\lambda-1))\\ &= q^{k+\lambda+2\mu-j}(G(k, \lambda-1, t, j-1, \mu; q)+G(k, \lambda-1, t, j, \mu; q)\\ &\qquad+G(k, \lambda-1, t, j-2, \mu-1; q)\\ &\qquad+G(k, \lambda-1, t, j-1, \mu-1; q)). \end{aligned} \tag{2.11}$$

Now we know that $g(k, \lambda, t, j, \mu; q)$ and $G(k, \lambda, t, j, \mu; q)$ are identical if either λ or μ is zero. Thus since they both satisfy the recurrence (2.8), we have by induction on λ that they are equal in general. Thus Lemma 2.1 is established.

LEMMA 2.2. *Let $\pi_1(k, \lambda, t, j; m; n)$ denote the number of partitions of n into m parts of the form $n=\sum_{i=1}^{2\lambda+1} f_i\cdot i$, where*

(2.12) $f_1+\cdots+f_{\lambda+1}\leqq k-\lambda+j$;

(2.13) $f_{\lambda+1}+\cdots+f_{2\lambda+1}=k-t$;

(2.14) $f_\nu+\cdots+f_{\nu+\lambda}\geqq k$ *for some* ν;

(2.15) $f_a>1$ *implies* $a=\lambda+1$.

Let

$$g_1(k, \lambda, t, j; x; q) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^m q^n \pi_1(k, \lambda, t, j; m; n).$$

Then

$$g_1(k, \lambda, t, j; x; q) = x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^\mu \sigma_\mu(\lambda) \sum_{r=0}^{j-\mu-t} \sigma_{\lambda-r}(\lambda).$$

Proof. By the definitions of π and π_1, it is clear that

$$\pi_1(k, \lambda, t, j; k+\mu; n)-\pi_1(k, \lambda, t, j-1; k+\mu; n) = \pi(k, \lambda, t, j, \mu; n).$$

Hence

$$\begin{aligned} g_1(k, \lambda, t, j; x; q)-g_1(k, \lambda, t, j-1; x; q) &= \sum_{\mu=0}^{\infty} x^{k+\mu} g(k, \lambda, t, j, \mu; q) \\ &= \sum_{\mu=0}^{\infty} x^{k+\mu} q^{(k-t)(\lambda+1)} \sigma_\mu(\lambda)\sigma_{\lambda+\mu+t-j}(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} g_1(k, \lambda, t, j; x; q) &= \sum_{r=\lambda-k}^{j}\sum_{\mu=0}^{\infty} x^{k+\mu} q^{(k-t)(\lambda+1)} \sigma_\mu(\lambda)\sigma_{\lambda+\mu+t-r}(\lambda) \\ &= x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^\mu \sigma_\mu(\lambda) \sum_{r=\mu+t}^{j} \sigma_{\lambda+\mu+t-r}(\lambda) \\ &= x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^\mu \sigma_\mu(\lambda) \sum_{r=0}^{j-\mu-t} \sigma_{\lambda-r}(\lambda). \end{aligned}$$

To conclude this section we prove a result concerning the multiplication of generating functions.

LEMMA 2.3. *Let $p_1(A; c; m, n)$ denote the number of partitions of n into m parts all $<c$ where the summands are subject to a set of conditions, A. Let $p_2(B; c; m, n)$ denote the number of partitions of n into m parts all $\geqq c$ where the summands are subject to a set of conditions, B. Let*

$$f_1(x, q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_1(A; c; m, n) x^m q^n,$$

$$f_2(x, q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_2(B; c; m, n) x^m q^n,$$

where we assume that $f_i(x, q)$ is absolutely convergent for $|x|<b_i$, $|q|<1$.

Then for $|x| < \min(b_1, b_2)$, $|q| < 1$, *if we let*

$$f_1(x, q)f_2(x, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_3(A, B; c; m, n)x^m q^n,$$

then $p_3(A, B; c; m, n)$ *is the number of partitions of* n *into* m *parts such that the summands* $< c$ *are subject to condition* A *and the summands* $\geqq c$ *are subject to condition* B.

Proof The result follows directly from

$$p_3(A, B; c; m, n) = \sum_{m_1+m_2=m;\, n_1+n_2=n} p_1(A; c; m_1, n_1)p_2(B; c; m_2, n_2).$$

3. **Proof of Theorem 1.** We begin by studing certain q-series considered in [3]. As in §2, we require that $k \geqq \max(1, 2\lambda - 1)$. We define

$$J_{\lambda,k,i}(x) = J_{\lambda,k,i}(x; q) = J_{k,i}(-q, -q^2, \ldots, -q^\lambda; x; q^{\lambda+1}); \tag{3.1}$$

$$H_{\lambda,k,i}(x) = H_{\lambda,k,i}(x; q) = H_{k,i}(-q, -q^2, \ldots, -q^\lambda; xq^{\lambda+1}; q^{\lambda+1}), \tag{3.2}$$

where (as in [3])

$$\begin{aligned} &J_{k,i}(a_1, \ldots, a_\lambda; x; q) \\ &= \sum_{n=0}^{\infty} (-1)^{n(\lambda+1)} x^{kn} (a_1 a_2 \cdots a_\lambda)^{-n} q^{(2k-\lambda+1)n^2/2 + (\lambda+1)n/2 + (k-i)n} \\ &\quad \cdot \prod_{r=1}^{n} (1-q^r)^{-1} \prod_{u=n+1}^{\infty} (1-xq^u)^{-1} \prod_{s=1}^{\lambda} \left(\prod_{v=0}^{n-1} (1-a_s q^v) \prod_{w=n+1}^{\infty} \left(1 - \frac{xq^w}{a_s}\right)\right) \\ &\quad \cdot \left\{1 + \frac{(-1)^{\lambda+1} x^i q^{(2n+1)i - \lambda n}(1-a_1 q^n)(1-a_2 q^n)\cdots(1-a_\lambda q^n)}{a_1 a_2 \cdots a_\lambda (1 - xq^{n+1}a_1^{-1})(1-xq^{n+1}a_2^{-1})\cdots(1-xq^{n+1}a_\lambda^{-1})}\right\}; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &H_{k,i}(a_1, \ldots, a_\lambda; x; q) \\ &= \sum_{n=0}^{\infty} (-1)^{n(\lambda+1)} x^{kn} (a_1 a_2 \cdots a_\lambda)^{-n} q^{(2k-\lambda+1)n^2/2 + (\lambda+1)n/2 - in}(1 - x^i q^{2ni}) \\ &\quad \cdot \prod_{r=1}^{n} (1-q^r)^{-1} \prod_{u=n}^{\infty} (1-xq^u)^{-1} \prod_{s=1}^{\lambda} \left(\prod_{v=0}^{n-1} (1-a_s q^v) \prod_{w=n+1}^{\infty} \left(1 - \frac{xq^w}{a_s}\right)\right). \end{aligned} \tag{3.4}$$

Then from [3, equation (2.1)], we have

$$H_{\lambda,k,i}(x) - H_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}(xq^{\lambda+1}); \tag{3.5}$$

from [3, equation (2.2)], we have

$$J_{\lambda,k,i}(x) = \sum_{j=0}^{\lambda} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x); \tag{3.6}$$

from [3, equation (2.3)], we have

$$H_{\lambda,k,-i}(x) = -(xq^{\lambda+1})^{-i} H_{\lambda,k,i}(x), \tag{3.7}$$

and from [3, equation (2.8)], we have

$$H_{\lambda,k,1}(x) = J_{\lambda,k,k}(xq^{\lambda+1}) = J_{\lambda,k,k+1}(xq^{\lambda+1}). \tag{3.8}$$

LEMMA 3.1. *Let $|q|<1$, and let $H^*_{\lambda,k,i}(x)$ $(0\leqq i\leqq k)$ and $J^*_{\lambda,k,i}(x)$ $(1\leqq i\leqq k)$ be any collection of functions analytic in x in the neighborhood of zero which satisfy*

$$H^*_{\lambda,k,i}(x)-H^*_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1}J^*_{\lambda,k,k-i+1}(xq^{\lambda+1}), \qquad 1\leqq i\leqq k; \tag{3.9}$$

$$H^*_{\lambda,k,0}(x) = 0; \tag{3.10}$$

$$\begin{aligned} J^*_{\lambda,k,i}(x) = &\sum_{j=0}^{\min(i,\lambda)} x^j\sigma_j(\lambda)H^*_{\lambda,k,i-j}(x) \\ &-x^i\sum_{j=\min(i,\lambda)+1}^{\lambda} q^{(\lambda+1)(i-j)}\sigma_j(\lambda)H^*_{\lambda,k,j-i}(x); \end{aligned} \tag{3.11}$$

$$H^*_{\lambda,k,i}(0) = J^*_{\lambda,k,i}(0) = 1, \qquad 1\leqq i\leqq k. \tag{3.12}$$

Then $H^*_{\lambda,k,i}(x)=H_{\lambda,k,i}(x)$, and $J^*_{\lambda,k,i}(x)=J_{\lambda,k,i}(x)$.

Proof. We need only show that solutions of the above equations are unique since (3.3)–(3.8) imply that the $H_{\lambda,k,i}(x)$ and $J_{\lambda,k,i}(x)$ fulfill the conditions of the lemma. If we can show that the $H^*_{\lambda,k,i}(x)$ are uniquely determined, then since the $J^*_{\lambda,k,i}(x)$ are defined in terms of the $H^*_{\lambda,k,i}(x)$ by (3.11), the $J^*_{\lambda,k,i}(x)$ will be uniquely determined also.

From (3.9) and (3.11) we have

$$\begin{aligned} &H^*_{\lambda,k,i}(x)-H^*_{\lambda,k,i-1}(x) \\ &= (xq^{\lambda+1})^{i-1}\sum_{j=0}^{\min(k-i+1,\lambda)} x^jq^{j(\lambda+1)}\sigma_j(\lambda)H^*_{\lambda,k,k-i+1-j}(xq^{\lambda+1}) \\ &\quad -(xq^{\lambda+1})^{k-i+1}\sum_{j=\min(k-i+1,\lambda)+1}^{\lambda} q^{(\lambda+1)(k-i+1-j)}\sigma_j(\lambda)H^*_{\lambda,k,j-k+i-1}(xq^{\lambda+1}). \end{aligned} \tag{3.13}$$

Hence if $H^*_{\lambda,k,i}(x)=\sum_{n=0}^{\infty}\eta_n(i)x^n$, then by (3.10) and (3.12)

$$\begin{aligned} \eta_0(i) &= 1, \quad 1\leqq i\leqq k, \\ &= 0, \quad i=0. \end{aligned}$$

From (3.13),

$$\begin{aligned} \eta_n(1)-\eta_n(k)q^{n(\lambda+1)} &= \text{polynomial in } \eta_j\text{'s with } j<n, \\ \eta_n(2)-\eta_n(1) &= \text{polynomial in } \eta_j\text{'s with } j<n, \\ &\vdots \\ \eta_n(k)-\eta_n(k-1) &= \text{polynomial in } \eta_j\text{'s with } j<n. \end{aligned}$$

We consider the above equations as a system of k equations in the k unknowns $\eta_n(1), \eta_n(2), \ldots, \eta_n(k)$. The determinant of the system is $1-q^{n(\lambda+1)} \neq 0$ since $|q|<1$. Hence the $\eta_n(i)$ are uniquely determined by the η_j's with $j<n$. Hence by mathematical induction the $\eta_n(i)$ are unique, and therefore the $H^*_{\lambda,k,i}(x)$ are unique. Thus Lemma 3.1 is proved.

LEMMA 3.2. *Suppose* $|q|<1$, $k \geqq \max(1, 2\lambda-1)$. *Let* $H^\#_{\lambda,k,i}(x)$, $0 \leqq i \leqq k$ *and* $J^\#_{\lambda,k,i}(x)$, $1 \leqq i \leqq k$ *be any collection of functions analytic in* x *in the neighborhood of zero which satisfy*

$$H^\#_{\lambda,k,i}(x) - H^\#_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1} J^\#_{\lambda,k,k-i+1}(xq^{\lambda+1}), \qquad 1 \leqq i \leqq k; \tag{3.14}$$

$$H^\#_{\lambda,k,0}(x) = 0; \tag{3.15}$$

$$J^\#_{\lambda,k,i}(x) = \sum_{j=0}^{\min(i,\lambda)} x^j \sigma_j(\lambda) H^\#_{\lambda,k,i-j}(x), \qquad 1 \leqq i \leqq k-\lambda+1; \tag{3.16}$$

$$\begin{aligned} J^\#_{\lambda,k,k-\lambda+1+j}(x) &= \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H^\#_{\lambda,k,k-\lambda+1+j-r}(x) \\ &\quad - \sum_{r=0}^{j} g_1(k, \lambda, r, j; x; q) H^\#_{\lambda,k,r}(xq^{\lambda+1}), \qquad 1 \leqq j < \lambda, \end{aligned} \tag{3.17}$$

where $g_1(k, \lambda, r, j; x; q)$ *is defined in Lemma* 2.2.

$$H^\#_{\lambda,k,i}(0) = 1, \quad 1 \leqq i \leqq k; \qquad J^\#_{\lambda,k,i}(0) = 1, \quad 1 \leqq i \leqq k. \tag{3.18}$$

Then if we define

$$\begin{aligned} H^*_{\lambda,k,i}(x) &= H^\#_{\lambda,k,i}(x), \qquad 0 \leqq i \leqq k-\lambda+1, \\ &= H^\#_{\lambda,k,i}(x) - \sum_{r=1}^{j} H^\#_{\lambda,k,r}(xq^{\lambda+1}) x^k q^{(\lambda+1)(k-r)} \sum_{m=0}^{j-r} \sigma_{\lambda-m}(\lambda), \\ &\qquad i = k-\lambda+1+j,\ 1 \leqq j < \lambda; \end{aligned} \tag{3.19}$$

$$J^*_{\lambda,k,i}(x) = x^{-k+i}(H^*_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H^*_{\lambda,k,k-i}(xq^{-\lambda-1})), \tag{3.20}$$

then $H^*_{\lambda,k,i}(x)$ *and* $J^*_{\lambda,k,i}(x)$ *fulfill the equations of Lemma* 3.1 *and for* $\lambda \leqq i \leqq k$, $J^*_{\lambda,k,i}(x) = J^\#_{\lambda,k,i}(x)$.

Proof. Obviously (3.9) and (3.10) are fulfilled from (3.20) and (3.15). Clearly for $\lambda \leqq i \leqq k$,

$$\begin{aligned} J^*_{\lambda,k,i}(x) &= x^{-k+i}(H^*_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H^*_{\lambda,k,k-i}(xq^{-\lambda-1})) \\ &= x^{-k+i}(H^\#_{\lambda,k,k-i+1}(xq^{-\lambda-1}) - H^\#_{\lambda,k,k-i}(xq^{-\lambda-1})) \\ &= J^\#_{\lambda,k,i}(x). \end{aligned}$$

Now if $\lambda \leqq i \leqq k-\lambda+1$, (3.19) and (3.16) show that (3.11) is valid. If $i=k-\lambda+1+j$, $1 \leqq j<\lambda$, then

$$\begin{aligned}
J^{*}_{\lambda,k,k-\lambda+1+j}(x) &= J^{\#}_{\lambda,k,k-\lambda+1+j}(x) \\
&= \sum_{r=0}^{j-1} x^r\sigma_r(\lambda)H^{\#}_{\lambda,k,k-\lambda+1+j-r}(x)+\sum_{r=j}^{\lambda} x^r\sigma_r(\lambda)H^{\#}_{\lambda,k,k-\lambda+1+j-r}(x) \\
&\quad -\sum_{r=0}^{j} g_1(k,\lambda,r,j;x;q)H^{\#}_{\lambda,k,r}(xq^{\lambda+1}) \\
&= \sum_{r=0}^{j-1} x^r\sigma_r(\lambda)\Bigg(H^{*}_{\lambda,k,k-\lambda+1+j-r}(x)+\sum_{v=1}^{j-r} H^{*}_{\lambda,k,v}(xq^{\lambda+1})(xq^{\lambda+1})^k q^{-v(\lambda+1)} \\
&\qquad\qquad \sum_{w=0}^{j-r-v} \sigma_{\lambda-w}(\lambda)\Bigg) \\
&\quad +\sum_{r=j}^{\lambda} x^r\sigma_r(\lambda)H^{*}_{\lambda,k,k-\lambda+1+j-r}(x) \\
&\quad -\sum_{r=0}^{j} g_1(k,\lambda,r,j;x;q)H^{*}_{\lambda,k,r}(xq^{\lambda+1}) \\
&= \sum_{r=0}^{\lambda} x^r\sigma_r(\lambda)H^{*}_{\lambda,k,k-\lambda+1+j-r}(x) \\
&\quad +\sum_{v=1}^{j} H^{*}_{\lambda,k,r}(xq^{\lambda+1})x^k q^{(k-v)(\lambda+1)} \sum_{r=0}^{j-v} x^r\sigma_r(\lambda) \sum_{w=0}^{j-r-v} \sigma_{\lambda-w}(\lambda) \\
&\quad -\sum_{r=0}^{j} g_1(k,\lambda,r,j;x;q)H^{*}_{\lambda,k,r}(xq^{\lambda+1}) \\
&= \sum_{r=0}^{\lambda} x^r\sigma_r(\lambda)H^{*}_{\lambda,k,k-\lambda+1+j-r}(x),
\end{aligned}$$

where the last equation follows by Lemma 2.2; thus again we have (3.11).

Finally, if $1 \leqq i<\lambda$, then

$$\begin{aligned}
J^{*}_{\lambda,k,i}(x) &= x^{-k+i}\Bigg(H^{\#}_{\lambda,k,k-i+1}(xq^{-\lambda-1})-\sum_{v=1}^{\lambda-i} H^{\#}_{\lambda,k,v}(x)x^k q^{-v(\lambda+1)} \sum_{w=0}^{\lambda-i-v} \sigma_{\lambda-w}(\lambda) \\
&\qquad -H^{\#}_{\lambda,k,k-i}(xq^{-\lambda-1})+\sum_{v=1}^{\lambda-i-1} H^{\#}_{\lambda,k,v}(x)x^k q^{-v(\lambda+1)} \sum_{w=0}^{\lambda-i-1-v} \sigma_{\lambda-w}(\lambda)\Bigg) \\
&= J^{\#}_{\lambda,k,i}(x)-x^i\sum_{v=1}^{\lambda-i} H^{\#}_{\lambda,k,v}(x)q^{-v(\lambda+1)}\sigma_{i+v}(\lambda) \\
&= \sum_{j=0}^{i} x^j\sigma_j(\lambda)H^{\#}_{\lambda,k,i-j}(x)-x^i\sum_{v=i+1}^{\lambda} H^{\#}_{\lambda,k,v-i}(x)q^{(i-v)(\lambda+1)}\sigma_v(\lambda) \\
&= \sum_{j=0}^{i} x^j\sigma_j(\lambda)H^{*}_{\lambda,k,i-j}(x)-x^i\sum_{v=i+1}^{\lambda} H^{*}_{\lambda,k,v-i}(x)q^{(i-v)(\lambda+1)}\sigma_v(\lambda),
\end{aligned}$$

which again is (3.11).

Now from (3.19) it is clear that $H^{*}_{\lambda,k,i}(0)=1$, $1\leqq i\leqq k$, and since $J^{*}_{\lambda,k,i}(x)$ satisfies (3.11), $J^{*}_{\lambda,k,i}(0)=1$, $1\leqq i\leqq k$. The analytic nature of these functions is now obvious from (3.19) and (3.11). Thus we have Lemma 3.2.

COROLLARY 3.3. *The set of functions described in the hypothesis of Lemma* 3.2 *is unique and exists.*

Proof. By Lemma 3.1, the $H^{*}_{\lambda,k,i}(x)$ and $J^{*}_{\lambda,k,i}(x)$ defined in Lemma 3.2 are unique, and thus by (3.19), (3.14), and (3.16) so must $H^{\#}_{\lambda,k,i}(x)$ and $J^{\#}_{\lambda,k,i}(x)$ be. For existence take $H^{\#}_{\lambda,k,i}(x)=H_{\lambda,k,i}(x)$, $0\leqq i\leqq k-\lambda+1$,

$$H^{\#}_{\lambda,k,i}(x) = H_{\lambda,k,i}(x)+\sum_{r=1}^{j} H_{\lambda,k,r}(xq^{\lambda+1})x^{k}q^{(\lambda+1)(k-r)}\sum_{m=0}^{j-r}\sigma_{\lambda-m}(\lambda),$$
$$i = k-\lambda+1+j,\ 1 \leqq j < \lambda,$$

$$J^{\#}_{\lambda,k,i}(x) = J_{\lambda,k,i}(x), \qquad \lambda \leqq i \leqq k,$$

$$J^{\#}_{\lambda,k,i}(x) = \sum_{j=0}^{i} x^{j}\sigma_{j}(\lambda)H_{\lambda,k,i-j}(x), \qquad 1 \leqq i < \lambda.$$

Thus we have the corollary.

We are now in a position to establish our main result.

Proof of Theorem 1. Let $P_{\lambda,k,a}(m,n)$, $1\leqq a\leqq k$, denote the number of partitions of n into m parts of the form $n=\sum_{i=1}^{\infty} f_i\cdot i$, where $f_i+\cdots+f_{i+\lambda}\leqq k-1$, $f_{i(\lambda+1)}+\cdots+f_{(i+1)(\lambda+1)}\leqq k-1$, $f_1+\cdots+f_{\lambda+1}\leqq a-1$, $f_b>1$ then $\lambda+1|b$ (equivalently, $n=b_1+\cdots+b_m$, $b_i\geqq b_{i+1}$, $b_i-b_{i+k-1}\geqq\lambda+1$ with strict inequality if $\lambda+1|b_i$, the total number of parts $\leqq\lambda+1$ does not exceed $a-1$, and only multiples of $\lambda+1$ may be repeated).

Let $Q_{\lambda,k,a}(m,n)$ denote the number of partitions described above in which $f_1=f_2=\cdots=f_\lambda=0$ (i.e., the smallest part is $\geqq\lambda+1$).

Let

$$J'_{\lambda,k,a}(x) = 1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{\lambda,k,a}(m,n)x^{m}q^{n}, \qquad 1 \leqq a \leqq k;$$

$$H'_{\lambda,k,a}(x) = 1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{\lambda,k,a}(m,n)x^{m}q^{n}, \qquad 1 \leqq a \leqq k;$$

$$H'_{\lambda,k,0}(x) = 0.$$

By comparison with the generating function for the number of ordinary partitions of n into m parts, we see that we have convergence for $|q|<1$, $|x|<|q|^{-1}$; thus, for given $|q|<1$, the above are analytic in a circle of radius >1 around $x=0$.

Also $H'_{\lambda,k,a}(0)=J'_{\lambda,k,a}(0)=1$, $1\leqq a\leqq k$.

Our next object is to show that these functions satisfy (3.14)–(3.17).

Beginning with (3.14), we consider $Q_{\lambda,k,i}(m,n)-Q_{\lambda,k,i-1}(m,n)$; this expression enumerates those partitions considered by $Q_{\lambda,k,i}(m,n)$ in which $\lambda+1$ appears exactly $i-1$ times. Let us now subtract $\lambda+1$ from each part in such a partition; the number being partitioned is reduced to $n-m(\lambda+1)$, and the number of parts appearing is now $m-i+1$. Since we had originally

$$f_{\lambda+2}+\cdots+f_{2\lambda+2} = k-1-f_{\lambda+1} = (k-i+1)-1,$$

we have after subtraction

$$f_1+\cdots+f_{\lambda+1} = (k-i+1)-1.$$

Otherwise the conditions on the summands are just shifted but unaltered. Thus we are now considering a partition of the type enumerated by

$$P_{\lambda,k,k-i+1}(m-i+1, n-m(\lambda+1)).$$

The above procedure establishes a one-to-one correspondence between the partitions enumerated by $Q_{\lambda,k,i}(m, n)-Q_{\lambda,k,i-1}(m, n)$ and those enumerated by $P_{\lambda,k,k-i+1}(m-i+1, n-m(\lambda+1))$. Hence

$$Q_{\lambda,k,i}(m, n)-Q_{\lambda,k,i-1}(m, n) = P_{\lambda,k,k-i+1}(m-i+1, n-m(\lambda+1)).$$

In terms of generating functions, this implies

$$H'_{\lambda,k,i}(x)-H'_{\lambda,k,i-1}(x) = (xq^{\lambda+1})^{i-1}J'_{\lambda,k,k-i+1}(xq^{\lambda+1}),$$

which is (3.14).

(3.15) is by definition.

We now treat (3.16). I claim that for $1 \leqq i \leqq k-\lambda+1$

$$x^j\sigma_j(\lambda)H'_{\lambda,k,i-j}(x), \qquad 0 \leqq j \leqq \min(i, \lambda), \tag{3.21}$$

is the generating function for the number of partitions of the type enumerated by $P_{\lambda,k,i}(m, n)$ in which exactly j parts $\leqq \lambda$ appear.

By Lemma 2.3, (3.21) is the generating function for partitions in which there are j distinct parts $\leqq \lambda$ and the parts $\geqq \lambda+1$ satisfy the desired conditions. Is it possible then that any of the conditions on the partitions enumerated by $P_{\lambda,k,i}(m, n)$ could be violated? This is not possible because for $1 \leqq v \leqq \lambda$

$$f_v+\cdots+f_\lambda \leqq j; \qquad f_{\lambda+1} \leqq i-j-1; \qquad f_{\lambda+2}+\cdots+f_{\lambda+v} \leqq v-1;$$

hence

$$f_v+\cdots+f_{v+\lambda} \leqq j+i-j-1+v-1 = i+v-2.$$

If $v=1$, this is $f_1+\cdots+f_{\lambda+1} \leqq i-1$ as desired, and if $1<v\leqq\lambda$, this is $f_v+\cdots+f_{v+\lambda} \leqq i+v-2 \leqq k-\lambda+1+\lambda-2=k-1$ as desired.

Conversely it is easily seen that any partition of the type enumerated by $P_{\lambda,k,i}(m, n)$ with exactly j parts $\leqq \lambda$ is of the type considered by (3.21).

Thus clearly

$$\begin{aligned} J'_{\lambda,k,i}(x) &= \sum_{j=0}^{i} x^j\sigma_j(\lambda)H'_{\lambda,k,i-j}(x) \\ &= \sum_{j=0}^{\min(i,\lambda)} x^j\sigma_j(\lambda)H'_{\lambda,k,i-j}(x), \qquad 1 \leqq i \leqq k-\lambda+1, \end{aligned}$$

which is (3.16).

For (3.17) we note that the above procedure must be modified since now in considering (3.21) with $i>k-\lambda+1$ it is indeed possible that there exists v with $1<v\leqq\lambda$, such that $f_v+\cdots+f_{v+\lambda}\geqq k$.

Indeed by the preceding argument for $1\leqq j<\lambda$

$$\sum_{r=0}^{\lambda} x^r\sigma_r(\lambda)H'_{\lambda,k,k-\lambda+1+j-r}(x)-J'_{\lambda,k,k-\lambda+1+j}(x) \tag{3.22}$$

is the generating function for partitions of the type enumerated by

$$P_{\lambda,k,k-\lambda+1+j}(m, n)$$

except for the fact that for some v, $1<v\leqq\lambda$, $f_v+\cdots+f_{v+\lambda}\geqq k$.

On the other hand, let us consider

$$g_1(k, \lambda, r, j; x; q)H'_{\lambda,k,r}(xq^{\lambda+1}), \qquad 0 \leqq r < \lambda. \tag{3.23}$$

I claim that this is the generating function for partitions of the type described in the preceding paragraph in which $f_{\lambda+1}+\cdots+f_{2\lambda+1}=k-r$.

By Lemma 2.3, Lemma 2.2 (and the fact that $H'_{\lambda,k,r}(xq^{\lambda+1})$ is the generating function for partitions of the type enumerated by $P_{\lambda,k,r}(m, n)$ with the added restriction that $f_1=f_2=\cdots=f_{2\lambda+1}=0$, $f_{2\lambda+2}\leqq r-1$), we see that (3.23) is the generating function for partitions of n into m parts of the form $n=\sum_{i=1}^{\infty} f_i\cdot i$, where

$$f_1+\cdots+f_{\lambda+1} \leqq k-\lambda+j; \tag{3.24}$$

$$f_{\lambda+1}+\cdots+f_{2\lambda+1} = k-r; \tag{3.25}$$

$$f_v+\cdots+f_{v+\lambda} \geqq k \quad \text{for some } v, \qquad 1 < v \leqq \lambda; \tag{3.26}$$

$$f_a > 1 \quad \text{implies } \lambda+1|a; \tag{3.27}$$

$$f_m+\cdots+f_{m+\lambda} \leqq k-1, \quad \text{for } m \geqq 2\lambda+2; \tag{3.28}$$

$$f_{m(\lambda+1)}+\cdots+f_{(m+1)(\lambda+1)} \leqq k-1, \qquad m \geqq 2; \tag{3.29}$$

$$f_{2\lambda+2} \leqq r-1. \tag{3.30}$$

We shall establish the claim made for (3.23) if we can show that the following two conditions hold for the partitions considered.

$$f_m+\cdots+f_{m+\lambda} \leqq k-1, \quad \text{for } \lambda+1 < m < 2\lambda+2; \tag{3.31}$$

$$f_{\lambda+1}+\cdots+f_{2\lambda+2} \leqq k-1. \tag{3.32}$$

Now (3.32) is clear from (3.25) and (3.30). As for (3.31) we have

$$f_m+\cdots+f_{2\lambda+1}+f_{2\lambda+3}+\cdots+f_{m+\lambda} \leqq \lambda; \quad f_{2\lambda+2} \leqq r-1;$$

therefore

$$f_m+\cdots+f_{m+\lambda} \leqq \lambda+r-1 \leqq \lambda+\lambda-2 = 2\lambda-2 \leqq k-1.$$

Thus (3.23) is the desired generating function.

Consequently for $1 \leqq j < \lambda$,

$$\begin{aligned}\sum_{r=0}^{\lambda} x^r\sigma_r(\lambda)H'_{\lambda,k,k-\lambda+1+j-r}(x)-J'_{\lambda,k,k-\lambda+1+j}(x) \\ &= \sum_{r=0}^{\lambda-1} g_1(k,\lambda,r,j;x;q)H'_{\lambda,k,r}(xq^{\lambda+1}) \\ &= \sum_{r=0}^{j} g_1(k,\lambda,r,j;x;q)H'_{\lambda,k,r}(xq^{\lambda+1}),\end{aligned}$$

which is (3.17); the second to last equation is summed to $\lambda-1$ only because (3.25) and (3.24) contradict (3.26) if $r \geqq \lambda$.

Thus by Corollary 3.3 and its proof,

$$J'_{\lambda,k,i}(x) = J_{\lambda,k,i}(x), \qquad \lambda \leqq i \leqq k. \tag{3.33}$$

Finally

$$\begin{aligned}1+\sum_{n=1}^{\infty} B_{\lambda,k,i}(n)q^n \\ &= 1+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P_{\lambda,k,i}(m,n)q^n \\ &= J_{\lambda,k,i}(1) \\ &= \prod_{j=1}^{\infty}(1+q^j)(1+q^{j(\lambda+1)})^{-1}(1-q^{j(\lambda+1)})^{-1}(1-q^{(2k-\lambda+1)(\lambda+1)j-(i-\frac{1}{2}\lambda)(\lambda+1)}) \\ &\quad \cdot(1-q^{(2k-\lambda+1)(\lambda+1)(j-1)+(i-\frac{1}{2}\lambda)(\lambda+1)})(1-q^{(2k-\lambda+1)(\lambda+1)j}) \\ &= \begin{cases} \displaystyle\prod_{\substack{j=1 \\ j\not\equiv 0 \pmod{\lambda+1}}}^{\infty}(1+q^j) \prod_{\substack{s=0 \\ s\equiv 0 \pmod{\lambda+1} \\ s\not\equiv 0,\,\pm(i-\frac{1}{2}\lambda)(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}}}^{\infty}(1-q^s)^{-1}, & \lambda \text{ even} \\ \displaystyle\prod_{\substack{j=1 \\ j\not\equiv 0 \pmod{\frac{1}{2}(\lambda+1)}}}^{\infty}(1+q^j) \prod_{\substack{s=0 \\ s\equiv 0,\frac{1}{2}(\lambda+1),\frac{3}{2}(\lambda+1) \pmod{2\lambda+2} \\ s\not\equiv 0,\,\pm(2i-\lambda)\frac{1}{2}(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}}}^{\infty}(1-q^s)^{-1}, & \lambda \text{ odd}\end{cases} \\ &= 1+\sum_{n=1}^{\infty} A_{\lambda,k,i}(n)q^n,\end{aligned}$$

where the third to last equation follows from [3, equation (2.9)]. Hence $A_{\lambda,k,i}(n) = B_{\lambda,k,i}(n)$ and Theorem 1 is established.

4. **Extension of Theorem 1.** We extend our definition of $B_{\lambda,k,a}(n)$ as follows. Let $B^*_{\lambda,k,a}(n)$ denote those partitions of n enumerated by $B_{\lambda,k,a}(n)$ such that for any integer j in the interval $[1, \frac{1}{2}(\lambda+1)]$, there must be at most $a-j$ parts of the partition in the interval $[j, \lambda-j+1]$ (in terms of frequency notation the new condition is $f_j+\cdots+f_{\lambda-j+1}\leqq a-j$, $1\leqq j\leqq \frac{1}{2}(\lambda+1)$). We define $P^*_{\lambda,k,a}(m,n)$ similarly.

We remark that for $a\geqq\lambda$, $B^*_{\lambda,k,a}(n)=B_{\lambda,k,a}(n)$. This is because the new conditions are automatically fulfilled for $a\geqq\lambda$, since the number of summands in the interval $[j, \lambda-j+1]$ is $\leqq \min(a-1, \lambda-2j+2)\leqq \min(a-1, a-2j+2)\leqq a-j$.

In Theorem 1 we note that $A_{\lambda,k,a}(n)$ is well defined for $\frac{1}{2}\lambda<a\leqq k$. We now have the following extension of Theorem 1.

THEOREM 2. *$A_{\lambda,k,a}(n)=B^*_{\lambda,k,a}(n)$ for $\frac{1}{2}\lambda<a\leqq k$, $k\geqq 2\lambda-1$.*

This theorem will be easy to deduce from Theorem 1 after we digress briefly for a study of a new auxiliary partition function.

Let $\varphi(i,j,\lambda;n)$, $\frac{1}{2}\lambda<i\leqq\lambda$, $i\leqq j\leqq\lambda$, be the number of partitions of n into $2i-j$ distinct parts of the form $n=\sum_{e=1}^{\lambda} f_e\cdot e$ where *at least one* of the following inequalities holds

$$(4.1)_b \qquad f_b+\cdots+f_{\lambda-b+1} > i-b, \qquad 1\leqq b\leqq \tfrac{1}{2}(\lambda+1).$$

Let $\psi(i,j,\lambda;q)=\sum_{n=0}^{\infty}\varphi(i,j,\lambda;n)q^n$.

LEMMA 4.1. *For $\frac{1}{2}\lambda<i\leqq\lambda$, $i\leqq j\leqq\lambda$,*

$$(4.2) \qquad \psi(i,j,\lambda;q) = q^{(i-j)(\lambda+1)}\sigma_j(\lambda).$$

Proof. Now if $i=j$, then $2i-j=i$. Therefore $f_1+\cdots+f_\lambda=i>i-1$. Hence $\psi(i,i,\lambda;q)$ is just the generating function for partitions into i distinct parts all $\leqq\lambda$. Thus

$$(4.3) \qquad \psi(i,i,\lambda;q) = \sigma_i(\lambda).$$

We now wish to show

$$(4.4) \qquad \psi(i+1,j,\lambda;q) = q^{\lambda+1}\psi(i,j,\lambda;q);$$

(4.4) together with (4.3) imply (4.2) by mathematical induction.

We proceed as follows. For a given partition, π, of the type enumerated by $\varphi(i,j,\lambda;n)$, define

$$(4.5) \qquad m(\pi) = \max_{1\leqq b\leqq\frac{1}{2}(\lambda+1)} f_b+\cdots+f_{\lambda-b+1}-i+b;$$

$$(4.6) \qquad U(\pi) = \min\{b \mid f_b+\cdots+f_{\lambda-b+1}-i+b = m(\pi)\};$$

$$(4.7) \qquad L(\pi) = \max\{b \mid f_b+\cdots+f_{\lambda-b+1}-i+b = m(\pi)\}.$$

We now note that

$$(4.8) \qquad f_{U(\pi)-1} = f_{\lambda-U(\pi)+2} = 0; \qquad f_{L(\pi)} = f_{\lambda-L(\pi)+1} = 1;$$

(4.8) is correct because the falsity of the first assertion would contradict the minimality of $U(\pi)$, and the falsity of the second assertion would contradict the maximality of $L(\pi)$ (note that $U(\pi)>1$ for $j>i$ because $i-1\geqq 2i-j=f_1+\cdots+f_\lambda$ in contrast with the fact that $(4.1)_b$ is valid for some b).

We now prove (4.4). If π is a partition enumerated by $\varphi(i,j,\lambda;n)$ delete from π the summands $L(\pi)$ and $\lambda-L(\pi)+1$, and call the new partition π'. π' is a partition of $n-\lambda-1$ with $2(i-1)-j$ parts.

$$U(\pi') = L(\pi)+1; \tag{4.9}$$

this is because if $b<L(\pi)+1$, then in π' we have

$$f_b'+\cdots+f_{\lambda-b+1}'-(i-1)+b = f_b+\cdots+f_{\lambda-b+1}-2-i+1+b \leqq m(\pi)-1,$$

while for $b\geqq L(\pi)+1$

$$f_b'+\cdots+f_{\lambda-b+1}'-(i-1)+b = f_b+\cdots+f_{\lambda-b+1}-i+1+b \leqq m(\pi),$$

and

$$\begin{aligned} f_{L(\pi)+1}'+\cdots+f_{\lambda-L(\pi)}'-(i-1)+L(\pi)+1 & \\ &= f_{L(\pi)}+\cdots+f_{\lambda-L(\pi)+1}-2-i+1+L(\pi)+1 \\ &= m(\pi). \end{aligned}$$

Now since originally $m(\pi)>0$, we see that $m(\pi)=m(\pi')>0$, and therefore π' is a partition of the type enumerated by $\varphi(i-1,j,\lambda;n-\lambda-1)$.

Conversely if π' is a partition of the type enumerated by $\varphi(i-1,j,\lambda;n-\lambda-1)$ then introduce the summands $U(\pi')-1$, and $\lambda-U(\pi')+2$ and call the new partition π''. π'' is a partition of n into $2i-j$ parts.

$$L(\pi'') = U(\pi')-1; \tag{4.10}$$

this is because if $b>U(\pi')-1$

$$\begin{aligned} f_b''+\cdots+f_{\lambda-b+1}''-i+b &= f_b'+\cdots+f_{\lambda-b+1}'-(i-1)+b-1 \\ &\leqq m(\pi')-1, \end{aligned}$$

while for $b\leqq U(\pi')-1$

$$\begin{aligned} f_b''+\cdots+f_{\lambda-b+1}''-i+b &= f_b'+\cdots+f_{\lambda-b+1}'-(i-1)+b+2-1 \\ &\leqq m(\pi')-1+2-1 = m(\pi'), \end{aligned}$$

and

$$\begin{aligned} f_{U(\pi')-1}''+\cdots+f_{\lambda-U(\pi')+2}''-i+U(\pi')-1 & \\ &= f_{U(\pi')}'+\cdots+f_{\lambda-U(\pi')+1}'+2-(i-1)+U(\pi')-2 \\ &= m(\pi'). \end{aligned}$$

Thus we see that the above procedure establishes a one-to-one correspondence between the partitions enumerated by $\varphi(i,j,\lambda;n)$ and those enumerated by $\varphi(i-1,j,\lambda;n-\lambda-1)$. Therefore

$$\varphi(i+1,j,\lambda;n) = \varphi(i,j,\lambda;n-\lambda-1). \tag{4.11}$$

We deduce (4.4) immediately from (4.11). Thus we have established Lemma 4.1.

Proof of Theorem 2. We have already noted that Theorem 2 is just Theorem 1 if $a \geqq \lambda$. We thus need only treat $\frac{1}{2}\lambda < a < \lambda$. Under these conditions we let $J''_{\lambda,k,a}(x)$ be the generating function for $P^*_{\lambda,k,a}(m, n)$. Then as in the proof of Theorem 1,

$$
(4.12)\quad \begin{aligned} J''_{\lambda,k,a}(x) &= J'_{\lambda,k,a}(x) - \sum_{j=a}^{\lambda} x^{2a-j}\psi(a, j, \lambda; q)H_{\lambda,k,j-a}(x) \\ &= \sum_{j=0}^{a} x^j\sigma_j(\lambda)H_{\lambda,k,a-j}(x) - \sum_{j=a+1}^{\lambda} x^{2a-j}\psi(a, j, \lambda; q)H_{\lambda,k,j-a}(x) \end{aligned}
$$

for the first term on the right-hand side (from the proof of Theorem 1) generates the partitions enumerated by $P_{\lambda,k,a}(m, n)$ and the second term, by the definition of ψ and Lemma 2.3, just removes those partitions enumerated by $P_{\lambda,k,a}(m, n)$ which are not enumerated by $P^*_{\lambda,k,a}(m, n)$.

Now on the other hand from (3.6) and (3.7)

$$
(4.13)\quad J_{\lambda,k,a}(x) = \sum_{j=0}^{a} x^j\sigma_j(\lambda)H_{\lambda,k,a-j}(x) - x^a \sum_{j=a+1}^{\lambda} q^{(a-j)(\lambda+1)}\sigma_j(\lambda)H_{\lambda,k,j-a}(x).
$$

Consequently by Lemma 4.1

$$
(4.14)\quad J''_{\lambda,k,a}(1) = J_{\lambda,k,a}(1).
$$

The remainder of the proof is now identical with that of Theorem 1.

5. **Conclusion.** Let us define $K(\lambda)$ to be that integer such that for $k \geqq K(\lambda)$ Theorem 2 (and consequently Theorem 1) holds.

CONJECTURE. $K(\lambda) = \lambda$.

It is fairly easy to show that $K(2) = 2$, and indeed numerical evidence indicates the truth of $K(\lambda) = \lambda$ for $\lambda = 3, 4, 5$. The main problem in proving the conjecture lies in the fact that the q-difference equations become much more involved for smaller values of k and the related partition identities become even more cumbersome.

A further interesting question seems to be whether or not even smaller values of k may be taken provided new restrictions are put on the partitions enumerated by $B_{\lambda,k,a}(n)$. As an example, Schur has proved [6, p. 495] that

$$
A_{3,2,2}(n) = B^0_{3,2,2}(n)
$$

where $B^0_{3,2,2}(n)$ denotes the number of partitions enumerated by $B_{3,2,2}(n)$ with the added restriction that no parts are $\equiv 2 \pmod 4$.

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